Formal Ontology in Information Systems
M. Donnelly and G. Guizzardi (Eds.)
IOS Press, 2012
© 2012 The authors and IOS Press. All rights reserved.
doi:10.3233/978-1-61499-084-0-177

Does your ontology make a (sense) difference?

Pawel Garbacz
The John Paul II Catholic University of Lublin

Abstract. The paper defines three logical criteria for semantic adequacy of an applied ontology. All criteria are based on the idea to the effect that when an ontology construed as a formal theory allows for swapping some items in its vocabulary, then it does not sufficiently differentiate between the meanings of these items and, consequently, the semantic aspect of this vocabulary cannot be claimed to be sufficiently characterised. Besides providing the formal definitions of those criteria and proving some simple correlations therebetween I present the empirical results of their implementation.

Keywords. semantic indeterminacy, ontology evaluation, meta-ontology

Applied formal ontology seems to be one of those peculiar disciplines that attempt to apply the methods of logic to solve the real-world problems of engineering. Therefore, if it is to live up to this expectation, it cannot be satisfied with producing complex and elegant abstract structures, but it needs to produce such structures for the sake of some application. Browsing through the mainstream papers in applied ontology, e.g., consulting [7] where eight main types of such applications are listed, one may get to the conclusion that one of the primary tasks of any applied formal ontology is to determine the semantic roles of its terminology, in particular to define the meanings of the terms, predicates, and (possibly) other symbols that occur therein. Since this task is usually interpreted as being attainable by means of determining classes of (semantic) models for those ontologies, what we can at best expect to achieve is *partial* clarifications and not adequate definitions. For example, the well-known model-theoretic limitations of first-order logic, like those described by Löwenheim-Skolem theorem, make it impossible to build a first-order theory whose class of models includes only intended models.

Even the modest task of partial semantic clarification may pose a challenge. In this paper I would like to define a package of three logical criteria that a formal applied ontology needs to satisfy if it is to achieve this task. Each criterion in this package is to be interpreted as a necessary condition: any ontology that does not meet it does not satisfactorily characterise its terminology. Since they can be ordered with respect to their logical strength, i.e., with respect to sets of ontologies that meet them, my recommendation can be interpreted as defining three grades of semantic indeterminacy without any further claim as to which of these grades defines a minimal threshold that a formal theory must reach to become an applied ontology worth of its name.

1. Insights and intuitions

Assume that a "toy" formal ontology $\mathfrak{O}_1$ consists of just one axiom:

$$\forall x[\mathsf{A}(x) \vee \mathsf{B}(x) \equiv \mathsf{C}(x)] \tag{1}$$

From the ontological point of view one can say that all this ontology achieves is that the category represented by predicate C is exhaustively divided into category (represented by) A and category (represented by) B. Obviously, $\mathfrak{O}_1$ adequately defines neither of these predicates – C may represent the category of numbers or the category of fatty acids while A and B may represent, respectively, even and odd numbers, or, unsaturated and saturated fatty acids. This feature however does not disqualifies $\mathfrak{O}_1$ as a means to semantic clarification. The more pertinent question is whether $\mathfrak{O}_1$ even partially characterises the meaning of its predicates or whether it characterises those meanings as specifically as possible. I claim that it does not because it does not differentiate between its predicates, i.e., it does not formally characterise them as separate categories. Since neither of the following formulae is a theorem of $\mathfrak{O}_1$:

$$\forall x[\mathsf{A}(x) \equiv \mathsf{C}(x)] \tag{2}$$

$$\forall x[\mathsf{A}(x) \equiv \mathsf{B}(x)] \tag{3}$$

$$\forall x[\mathsf{B}(x) \equiv \mathsf{C}(x)] \tag{4}$$

one is justified in believing that the three predicates have different meanings, but this difference is not encoded in the ontology itself.

First, you can swap A and B and $\mathfrak{O}_1$ will not change at all, i.e., no new theorem will be added and no old theorem will be removed therefrom. Thus, as far as $\mathfrak{O}_1$ is concerned A can be identical to B, i.e., there is no axiom in this ontology that states to the contrary.

Secondly, if you swap A and C, you will get a different ontology, say $\mathfrak{O}_2$, that contains the following axiom:

$$\forall x[\mathsf{C}(x) \vee \mathsf{B}(x) \equiv \mathsf{A}(x)] \tag{5}$$

Notice however that although 5 does not follow from 1, it is not inconsistent therewith either, i.e., you can add the former to ontology $\mathfrak{O}_1$ without making it inconsistent. Again, as far as $\mathfrak{O}_1$ is concerned A can be identical to C, i.e., there is no axiom in this ontology that states to the contrary. Metaphorically speaking, this "identity" is much weaker than the previous one since it is based on the consistency of swapping. In other words, one may say that the level of indeterminacy of the semantic difference between A and C (in this case) is lower than the level with respect to A and B (from the first case).

Thirdly, imagine now that we add axiom 6 to $\mathfrak{O}_1$ and obtain ontology $\mathfrak{O}_3$:

$$\neg\exists x\ [\mathsf{A}(x) \wedge \mathsf{B}(x)] \tag{6}$$

You may still swap A and B within $\mathfrak{O}_3$ and 5 is still consistent with $\mathfrak{O}_3$. However, when you add 5 to this extended ontology, you will get the following consequence:

$$\neg\exists x\ \mathsf{B}(x) \tag{7}$$

So although set $\{1, 6, 5\}$ of axioms is not inconsistent, one of its consequences has it that the extension of predicate/category B is empty. Since 7 does not follow from $\{1, 6\}$, one can assume that ontology $\mathfrak{O}_3$ does not characterise B as a null category (e.g., as `owl:Nothing`). In a sense one may claim that $\mathfrak{O}_3$ does not make room for 5 - not on the pain of inconsistency but because 5 would distort the ontological status of one of its categories. Therefore, it is not the case now that as far as $\mathfrak{O}_3$ is concerned A can be identical to C. From the intuitive point of view, this kind of semantic (in)determinacy can be located "between" the first and the second kind.

In sum, the three grades of semantic indeterminacy of an applied ontology can be intuitively characterised as below:

1. after swapping some of its terminology, the new axioms that result from swapping do not make the ontology inconsistent;
2. after swapping some of its terminology, the new axioms that result from swapping "do not add" to this ontology any empty categories, i.e., do not add any fact to the effect that a category from this ontology is empty;
3. after swapping some of its terminology, the ontology (construed as a formal theory) does not change at all.

2. Definitions

Let $\mathfrak{Lang}$ be a first-order language whose signature contains set $\mathfrak{Pred}$ of predicates. If $\delta \in \mathfrak{Pred}$, then $\mathfrak{ar}(\delta)$ gives the arity of predicate δ. The (meta-logical) symbol "$\delta(\vec{\gamma})$" is to denote an atomic formula of $\mathfrak{Lang}$ that is build from predicate δ and its arguments $\vec{\gamma}$. Similarly, "$\forall\vec{\gamma}\ \delta(\vec{\gamma})$" and "$\exists\vec{\gamma}\ \delta(\vec{\gamma})$" are, respectively, the full universal and existential closures of "$\delta(\vec{\gamma})$". The (meta-logical) function symbol "$\mathfrak{pred}$" will be used to find all predicates that occur in formulae from a given subset of $\mathfrak{Lang}$.

Now if $\Delta \subseteq \mathfrak{Pred}$, "$\sigma(\Delta)$" will denote a permutation on Δ, i.e., σ is a bijection from Δ onto Δ. Suppose that $\phi \in \mathfrak{Lang}$ and that $\emptyset \neq \Delta \subseteq \mathfrak{Pred}$. Suppose also that Δ is arity homogenous, i.e., if $\delta_1, \delta_2 \in \Delta$, then $\mathfrak{ar}(\delta_1) = \mathfrak{ar}(\delta_2)$. If $\sigma(\Delta)$ is a non-trivial permutation on Δ, i.e., if σ is not the identity relation in Δ, then $\mathsf{swap}_{\sigma(\Delta)}(\phi)$ is an element of $\mathfrak{Lang}$ that satisfies the following conditions:

1. if $\phi = \delta(\vec{\gamma})$ and $\delta \in \Delta$, then $\mathsf{swap}_{\sigma(\Delta)}(\phi) = \delta'(\vec{\gamma})$, where $\delta' = \sigma(\delta)$;
2. if $\phi = \delta(\vec{\gamma})$ and $\delta \notin \Delta$, then $\mathsf{swap}_{\sigma(\Delta)}(\phi) = \delta(\vec{\gamma})$;
3. if $\phi = \neg\psi$, then $\mathsf{swap}_{\sigma(\Delta)}(\phi) = \neg\mathsf{swap}_{\sigma(\Delta)}(\psi)$;
4. if $\phi = \forall\alpha\ \psi$, then $\mathsf{swap}_{\sigma(\Delta)}(\phi) = \forall\alpha\ \mathsf{swap}_{\sigma(\Delta)}(\psi)$.[1]

Note that $\mathsf{swap}_{\sigma(\Delta)}(\phi)$ is defined only if $|\Delta| > 1$ because otherwise $\sigma(\Delta)$ would be the identity in Δ.

For the sake of parsimony I will use the same symbol for a set of swapped formulae, i.e.,

$$\mathsf{swap}_{\sigma(\Delta)}(\Phi) \triangleq \{\psi : \exists\phi \in \Phi\ \psi = \mathsf{swap}_{\sigma(\Delta)}(\phi)\}. \tag{8}$$

Any language $\mathfrak{Lang}$ may be associated with a number consequence operations, which can be defined either semantically or proof-theoretically or in some other way (cf. [12]). Since the majority of existing formal ontologies are based on classical logic, $\mathfrak{C}$ is

[1] The conditions for other connectives are inferable from those given above.

here fixed as the consequence operation of first-order classical logic for language $\mathfrak{Lang}$. Consequently, I assume that $\mathfrak{C}$ has the following properties:

$$X \subseteq \mathfrak{C}(X) \tag{9}$$

$$X \subseteq Y \to \mathfrak{C}(X) \subseteq \mathfrak{C}(Y) \tag{10}$$

$$\mathfrak{C}(\mathfrak{C}(X)) \subseteq \mathfrak{C}(X) \tag{11}$$

$$\mathfrak{C}(\emptyset) \text{ contains all first-order tautologies in } \mathfrak{Lang} \tag{12}$$

$$\begin{aligned} &\phi \in \mathfrak{C}(\{\psi_1, \psi_2, \dots, \psi_n\}) \equiv \\ &\equiv \ulcorner \psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_n \to \phi \urcorner \in \mathfrak{C}(\emptyset) \end{aligned} \tag{13}$$

A *formal ontology* $\mathfrak{O}$ will be understood here as a couple $< \mathfrak{Lang}, \mathfrak{Ax} >$, where $\emptyset \neq \mathfrak{Ax} \subseteq \mathfrak{Lang}$ is a (finite) set of axioms (of this ontology). In what follows I will consider only consistent ontologies, i.e., those for which it is the case that

$$\mathfrak{C}(\mathfrak{Ax}) \neq \mathfrak{Lang} \tag{14}$$

A formal ontology $< \mathfrak{Lang}, \mathfrak{Ax} >$ will be called *empty* if for every $\delta \in \mathfrak{pred}(\mathfrak{Ax})$, $\ulcorner \neg \exists \vec{\gamma}\, \delta(\vec{\gamma}) \urcorner \in \mathfrak{C}(\mathfrak{Ax})$.

I will say that set $\Delta \subseteq \mathfrak{Pred}$ of predicates is *swappable* with respect to formal ontology $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ if $\Delta \subseteq \mathfrak{pred}(\mathfrak{Ax})$ and for any two predicates $\delta_1, \delta_2 \in \Delta$,

$$\ulcorner \forall \alpha [\delta_1(\alpha) \equiv \delta_2(\alpha)] \urcorner \notin \mathfrak{C}(\mathfrak{Ax}).$$

Now I will define the three grades of semantic indeterminacy, which I informally described in the previous section.

Definition 1. *A formal ontology* $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ exhibits the first grade of semantic indeterminacy *with respect to predicates from set* Δ *if and only if* Δ *is swappable with respect to* $\mathfrak{O}$ *and for every mapping* σ *it holds that*

$$\mathfrak{C}(\mathfrak{Ax} \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})) \neq \mathfrak{Lang}.$$

Definition 2. *A formal ontology* $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ exhibits the second grade of semantic indeterminacy *with respect to predicates from set* Δ *if and only if* Δ *is swappable with respect to* $\mathfrak{O}$ *and for every mapping* σ *and for every formula* $\phi = \ulcorner \neg \exists \vec{\gamma} \quad \delta(\vec{\gamma}) \urcorner$ *from* $\mathfrak{Lang}$ *it holds that*

$$\phi \notin \mathfrak{C}(\mathfrak{Ax}) \to \phi \notin \mathfrak{C}(\mathfrak{Ax} \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})).$$

Definition 3. *A formal ontology* $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ exhibits the third grade of semantic indeterminacy *with respect to predicates from set* Δ *if and only if* Δ *is swappable with respect to* $\mathfrak{O}$ *and for every mapping* σ *it holds that*

$$\mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax}) \subseteq \mathfrak{C}(\mathfrak{Ax}).$$

I presuppose that for each definition the respective "indeterminacy" condition is non-vacuously satisfied, i.e., for each of them there exists a mapping σ that satisfies the respective condition. Consequently, a formal ontology exhibits the first (*resp.* second, third) grade of semantic indeterminacy with respect to set Δ only if $|\Delta| > 1$.

3. Observations

It should be obvious that the three "toy" ontologies mentioned in section 1 exhibits the following grades of semantic indeterminacy:

1. $\mathfrak{O}_1$ exhibits all three grades with respect to $\{\mathsf{A},\mathsf{B}\}$, but only the first and second grade with respect to $\{\mathsf{A},\mathsf{C}\}$;
2. $\mathfrak{O}_2$ exhibits all three grades with respect to $\{\mathsf{B},\mathsf{C}\}$, but only the first and second grade with respect to $\{\mathsf{A},\mathsf{B}\}$;
3. $\mathfrak{O}_3$ exhibits all three grades with respect to $\{\mathsf{A},\mathsf{B}\}$, but only the first grade with respect to $\{\mathsf{A},\mathsf{C}\}$.

Now I will show the the order of the three grades of semantic indeterminacy can be established formally.

Fact 1. *If formal ontology $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ exhibits the third grade of semantic indeterminacy with respect to set Δ, then it also exhibits the second grade (with respect to Δ).*

Proof. Suppose that $\mathfrak{O}$ exhibits the third grade of semantic indeterminacy. Consequently, for every $\mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})$ it holds that

$$\mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax}) \subseteq \mathfrak{C}(\mathfrak{Ax}) \tag{†}$$

Now if $\mathfrak{O}$ does not exhibit the second grade of semantic indeterminacy, it means that there is a formula $\phi \in \mathfrak{Lang}$ such that

$$\phi \notin \mathfrak{C}(\mathfrak{Ax})$$

and

$$\phi \in \mathfrak{C}(\mathfrak{Ax} \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})).$$

But this latter consequence is inconsistent with the former given † and the following property of consequence operations:

$$X \subseteq \mathfrak{C}(Y) \rightarrow \mathfrak{C}(X \cup Y) \subseteq \mathfrak{C}(Y).$$

□

Fact 2. *If formal ontology $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ exhibits the second grade of semantic indeterminacy with respect to set Δ, then it also exhibits the first grade (with respect to Δ).*

Proof. Suppose that $\mathfrak{O}$ exhibits the second grade of semantic indeterminacy. Consequently, for every $\mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})$ and for every $\phi = \ulcorner \neg\exists\vec{\gamma} \quad \delta(\vec{\gamma})\urcorner$ from $\mathfrak{Lang}$ it holds that

$$\phi \in \mathfrak{C}(\mathfrak{Ax} \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})) \rightarrow \phi \in \mathfrak{C}(\mathfrak{Ax}) \tag{†}$$

Now if $\mathfrak{O}$ does not exhibit the first grade of semantic indeterminacy, it means that

$$\mathfrak{C}(\mathfrak{Ax} \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})) = \mathfrak{Lang} \tag{‡}$$

Pick up now predicate δ for which $\ulcorner \neg\exists\delta(\vec{\gamma})\urcorner \notin \mathfrak{C}(\mathfrak{Ax})$ - you can always find it as $\mathfrak{O}$ is not empty because of fact 6 below. Since $\ulcorner \neg\exists\delta(\vec{\gamma})\urcorner \in \mathfrak{Lang}$, ‡ together with † imply an inconsistency. □

Fact 3. *If formal ontology $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ exhibits the third grade of semantic indeterminacy with respect to set Δ, then it also exhibits the first grade (with respect to Δ).*

Proof. One can give a proof of this fact that is independent from facts 1 and 2. Suppose that $\mathfrak{O}$ exhibits the third grade of semantic indeterminacy. Consequently, for every $\mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})$ it holds that

$$\mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax}) \subseteq \mathfrak{C}(\mathfrak{Ax}) \tag{†}$$

Now if $\mathfrak{O}$ does not exhibit the first grade of semantic indeterminacy, it means that

$$\mathfrak{C}(\mathfrak{Ax} \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})) = \mathfrak{Lang} \tag{‡}$$

On the basis of the principle mentioned in the proof of fact 1 it follows that $\mathfrak{Lang} = \mathfrak{C}(\mathfrak{Ax} \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})) \subseteq \mathfrak{C}(\mathfrak{Ax})$. But this consequence is inconsistent with restriction 14. □

The particular observations made in the beginning of this section show that the opposite inclusions are not true. Thus, I will say that formal ontology $\mathfrak{O}$ exhibits the *proper first* grade of semantic indeterminacy (with respect to Δ) if $\mathfrak{O}$ exhibits the first, but not the second grade of semantic indeterminacy (with respect to Δ). Similarly, one may speak about proper second grade.

Now I will show that semantic indeterminacy is inheritable from "bigger" to "smaller" ontologies (fact 4) and from "bigger" to "smaller" sets of predicates (fact 5).

Fact 4. *If formal ontology $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ exhibits the first* (resp. *second, third) grade of semantic indeterminacy with respect to set Δ and $\mathfrak{Ax}' \subseteq \mathfrak{Ax}$, then it also exhibits the same grade with respect to $\Delta \cap \mathfrak{pred}(\mathfrak{Ax}')$ provided that* $|\Delta \cap \mathfrak{pred}(\mathfrak{Ax}')| > 1$.

Proof. Assume that $\mathfrak{O}$ exhibits the first (*resp.* second, third) grade of semantic indeterminacy with respect to set Δ and $\mathfrak{Ax}' \subseteq \mathfrak{Ax}$. Now pick up such permutation σ that $\mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})$ exists and such that the restriction σ' of σ to set $\Delta \cap \mathfrak{pred}(\mathfrak{Ax}')$ is a nontrivial permutation of the latter set. Since $\Delta \cap \mathfrak{pred}(\mathfrak{Ax}')$ is not a singleton, one can always find such a permutation. It should be obvious that

$$\mathsf{swap}_{\sigma'(\Delta \cap \mathfrak{pred}(\mathfrak{Ax}'))}(\mathfrak{Ax}') \subseteq \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax}), \tag{†}$$

Then one can easily establish fact 4 by inspection of the above definitions of grades given property 10 of the consequence operation. □

Fact 5. *If formal ontology* $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ *exhibits the first (*resp. *second, third) grade of semantic indeterminacy with respect to set* Δ *and* $\Delta' \subseteq \Delta$*, then it also exhibits the same grade with respect to* Δ' *provided that* $|\Delta'| > 1$.

Proof. First note that if $\Delta' \subseteq \Delta$ and $|\Delta'| > 1$, then for each permutation $\sigma'(\Delta')$ there exists a permutation $\sigma(\Delta)$ such that $\sigma'(\Delta') = \sigma(\Delta')$ and for each $\delta \in \Delta \setminus \Delta'$ $\sigma(\delta) = \delta$. Thus,

$$\mathsf{swap}_{\sigma'(\Delta')}(\mathfrak{Ax}) \subseteq \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax}) \tag{†}$$

As in the previous proof, this inclusion implies fact 5. □

Even if your ontology exhibits some grade of semantic indeterminacy as defined above, there is still some hope for you and your ontological artefact as you can always remove any type of indeterminacy by extending this ontology by additional axioms. However, in general only the first grade of semantic indeterminacy is easily removable - see fact 7 below - and the second and third grades may require substantial changes in your ontology - see facts 8 and 9. Oddly enough, all empty ontologies are free of any kind of indeterminacy.

Fact 6. *No empty ontology exhibits the first (*resp. *second, third) type of semantic indeterminacy with respect to any set* Δ *of predicates.*

Proof. The rationale for this fact is rather trivial. If formal ontology $< \mathfrak{Lang}, \mathfrak{Ax} >$ is empty, then for any predicates $\delta_1, \delta_2 \in \mathfrak{pred}(\mathfrak{Ax})$, it holds that $\ulcorner \forall \alpha [\delta_1(\alpha) \equiv \delta_2(\alpha)] \urcorner \in \mathfrak{C}(\mathfrak{Ax})$, which makes them unswappable. □

Fact 7. *Every formal ontology that exhibits the proper first grade of semantic indeterminacy can be conservatively extended to a formal ontology that does not exhibit the first grade, i.e., for every formal ontology* $< \mathfrak{Lang}, \mathfrak{Ax} >$ *that exhibits the proper first grade of semantic indeterminacy with respect to set* Δ*, there exists formal ontology* $< \mathfrak{Lang}, \mathfrak{Ax}' >$ *(*$\mathfrak{Ax} \subseteq \mathfrak{Ax}'$*) that does not exhibit the first grade of semantic indeterminacy with respect to* Δ *and such that* $\mathfrak{pred}(\mathfrak{Ax}') = \mathfrak{pred}(\mathfrak{Ax})$.

Proof. (The proof is similar to the standard proof of Lindenbaum's Lemma - [2, p. 26]). Assume that $\mathfrak{O} =< \mathfrak{Lang}, \mathfrak{Ax} >$ exhibits the first grade of semantic indeterminacy with respect to set Δ. This means that there is at least one mapping σ such that $\mathfrak{C}(\mathfrak{Ax} \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})) \neq \mathfrak{Lang}$. Moreover, the number of such mappings is finite, say is equal to n, and the set of formulae $\mathsf{swap}_{\sigma_i(\Delta)}(\mathfrak{Ax})$ that mapping σ_i determines is also finite, i.e., is equal to $|\mathfrak{Ax}|$. One can order all such formulae in a sequence $\phi_1, \phi_2, \ldots, \phi_k$, where $k \triangleq n * |\mathfrak{Ax}|$, on the basis of some syntactic feature of theirs. Then, following the standard proof of Lindenbaum's lemma, one can define a finite sequence $\mathfrak{Ax}_0, \mathfrak{Ax}_1, \ldots, \mathfrak{Ax}_k$ extension of $\mathfrak{Ax}$ as follows:

$$\mathfrak{Ax}_0 \triangleq \mathfrak{Ax}.$$

$$\mathfrak{Ax}_{m+1} \triangleq \begin{cases} \mathfrak{Ax}_m \cup \{\ulcorner \neg \phi_m \urcorner\} & \text{if } \mathfrak{Ax}_m \cup \{\ulcorner \neg \phi_m \urcorner\} \text{ is consistent;} \\ \mathfrak{Ax}_m & \text{otherwise.} \end{cases}$$

Consider now $\mathfrak{Ax}_k$. It is a finite and consistent set (with respect to $\mathfrak{C}$). Moreover, $\mathfrak{pred}(\mathfrak{Ax}_k) = \mathfrak{pred}(\mathfrak{Ax})$ because $\Delta \subseteq \mathfrak{pred}(\mathfrak{Ax})$, so for each mapping σ, $\mathfrak{pred}(\mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax})) \subseteq \mathfrak{pred}(\mathfrak{Ax})$. Thus, consider formal ontology $< \mathfrak{Lang}, \mathfrak{Ax}_k >$. Since $\mathfrak{O}$ exhibits the proper first grade of semantic indeterminacy, $\mathfrak{Ax} \subset \mathfrak{Ax}_k$. Then it is easy to observe that it does not exhibit the first grade of semantic indeterminacy because of our way of construction of $\mathfrak{Ax}_k$ and because of 12 and 13. $\square$

Fact 8. *For every formal ontology $< \mathfrak{Lang}, \mathfrak{Ax} >$ that exhibits the second grade of semantic indeterminacy with respect to set Δ, there exists a formal ontology $< \mathfrak{Lang}, \mathfrak{Ax}' >$ ($\mathfrak{Ax} \subseteq \mathfrak{Ax}'$) that does not exhibit the second grade of semantic indeterminacy with respect to Δ provided that $|\mathfrak{Pred} \setminus \mathfrak{pred}(\mathfrak{Ax})| > 1$.*

Proof. Consider two formulae:

$$\forall \vec{\gamma}[\delta'(\vec{\gamma}) \equiv \delta_1(\vec{\gamma}) \wedge \neg\delta_2(\vec{\gamma})] \tag{†}$$

$$\forall \vec{\gamma}[\delta''(\vec{\gamma}) \equiv \delta_2(\vec{\gamma}) \wedge \neg\delta_1(\vec{\gamma})] \tag{‡}$$

where $\delta_1, \delta_2 \in \Delta$ and both predicates δ' and δ'' ($\delta' \neq \delta''$) belongs to $\mathfrak{Pred} \setminus \mathfrak{pred}(\mathfrak{Ax})$. I claim that it *not* the case that *both*

$$\ulcorner \neg\exists \vec{\gamma}\, \delta'(\vec{\gamma}) \urcorner \in \mathfrak{C}(\mathfrak{Ax} \cup \{†\}) \tag{$\star$}$$

and

$$\ulcorner \neg\exists \vec{\gamma}\, \delta''(\vec{\gamma}) \urcorner \in \mathfrak{C}(\mathfrak{Ax} \cup \{‡\}) \tag{$\star\star$}$$

Assume otherwise. Then, because of $\star$, we would get that $\ulcorner \forall \vec{\gamma}[\delta_1(\vec{\gamma}) \rightarrow \delta_2(\vec{\gamma})] \urcorner \in \mathfrak{C}(\mathfrak{Ax})$ and, because of $\star\star$, $\ulcorner \forall \vec{\gamma}[\delta_2(\vec{\gamma}) \rightarrow \delta_1(\vec{\gamma})] \urcorner \in \mathfrak{C}(\mathfrak{Ax})$. But these two consequences contradict an assumption of the proof to the effect that δ_1 and δ_2 are swappable.

Suppose that $\ulcorner \neg\exists \vec{\gamma}\, \delta'(\vec{\gamma}) \urcorner \notin \mathfrak{C}(\mathfrak{Ax} \cup \{†\})$. Let $\mathfrak{Ax}' \triangleq \mathfrak{Ax} \cup \{†\}$. Since $\delta' \notin \mathfrak{pred}(\mathfrak{Ax})$, $\mathfrak{Ax}'$ is consistent with respect to $\mathfrak{C}$. That $< \mathfrak{Lang}, \mathfrak{Ax}' >$ does not exhibit the second grade of semantic indeterminacy follows from

$$\ulcorner \neg\exists \vec{\gamma}\, \delta'(\vec{\gamma}) \urcorner \in \mathfrak{C}(\mathfrak{Ax}' \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax}')) \tag{§}$$

for any mapping σ such that $\sigma(\delta_1) = \delta_2$ and $\sigma(\delta_2) = \delta_1$. Then $\mathfrak{C}(\mathfrak{Ax}' \cup \mathsf{swap}_{\sigma(\Delta)}(\mathfrak{Ax}'))$ contains also $\forall \vec{\gamma}[\delta'(\vec{\gamma}) \equiv \delta_2(\vec{\gamma}) \wedge \neg\delta_1(\vec{\gamma})]$ (besides †). Consequently, § holds. A similar argument can be made for the case when $\ulcorner \neg\exists \vec{\gamma}\, \delta''(\vec{\gamma}) \urcorner \notin \mathfrak{C}(\mathfrak{Ax} \cup \{‡\})$. $\square$

Notice that in general one cannot conservatively remove the second grade of semantic indeterminacy. That is to say, it is not the case that any formal ontology that exhibits the second grade of semantic indeterminacy with respect to set Δ can be conservatively extended to a formal ontology that does not exhibit it with respect to the same set. Consider for instance ontology $\mathfrak{O}_4$ whose set $\mathfrak{Ax}_4$ of axioms consists of just two formulae:

$$\exists x\, \mathsf{A}(x) \tag{15}$$

$$\exists x\ \mathsf{B}(x) \tag{16}$$

It is easy to see that $\mathfrak{O}_4$ exhibits the second (and also the third, for that matter) grade of semantic indeterminacy with respect to $\{\mathsf{A},\mathsf{B}\}$. Suppose that we conservatively extended $\mathfrak{Ax}_4$ with some additional axioms and arrive at an ontology whose set of axioms is $\mathfrak{Ax}'_4$. The term "conservatively" means that $\mathfrak{pred}(\mathfrak{Ax}'_4) = \{\mathsf{A},\mathsf{B}\}$. I claim that since $\mathfrak{Ax}'_4$ is consistent, there exists no formula $\ulcorner\neg\exists\vec{\gamma}\ \ \delta(\vec{\gamma})\urcorner$ such that it belongs to $\mathfrak{C}(\mathfrak{Ax}'_4 \cup \mathsf{swap}_{\sigma(\{\mathsf{A},\mathsf{B}\})}(\mathfrak{Ax}'_4))$ for any mapping σ on set $\{\mathsf{A},\mathsf{B}\}$. The reason is if there did, then δ would be an element of $\mathfrak{pred}(\mathfrak{Ax}'_4 \cup \mathsf{swap}_{\sigma(\{\mathsf{A},\mathsf{B}\})}(\mathfrak{Ax}'_4)) = \mathfrak{pred}(\mathfrak{Ax}'_4) = \{\mathsf{A},\mathsf{B}\}$ and this is impossible due to the consistency of $\mathfrak{Ax}'_4$. Finally, definition 2 implies then that the extended ontology based on $\mathfrak{Ax}'_4$ exhibits the second grade of semantic indeterminacy.

Fact 9. *For every formal ontology $< \mathfrak{Lang}, \mathfrak{Ax} >$ that exhibits the third grade of semantic indeterminacy with respect to set Δ, there exists a formal ontology $< \mathfrak{Lang}, \mathfrak{Ax}' >$ ($\mathfrak{Ax} \subseteq \mathfrak{Ax}'$) that does not exhibit the third grade of semantic indeterminacy with respect to Δ provided that $\mathfrak{Pred} \setminus \mathfrak{pred}(\mathfrak{Ax}) \neq \emptyset$.*

Proof. To show fact 9 one can appropriately accommodate the previous proof and obtain a simpler version thereof. We can now consider only one formula

$$\forall\vec{\gamma}[\delta'(\vec{\gamma}) \equiv \delta_1(\vec{\gamma}) \wedge \neg\delta_2(\vec{\gamma})] \tag{$\dagger$}$$

and show that

$$\ulcorner\neg\exists\vec{\gamma}\ \delta'(\vec{\gamma})\urcorner \notin \mathfrak{C}(\mathfrak{Ax} \cup \{\dagger\}) \tag{$\star$}$$

Assume otherwise. If formula $\ulcorner\neg\exists\vec{\gamma}\ \delta'(\vec{\gamma})\urcorner$ belonged to $\mathfrak{C}(\mathfrak{Ax} \cup \{\dagger\})$, this would imply not only that $\ulcorner\forall\vec{\gamma}[\delta_1(\vec{\gamma}) \rightarrow \delta_2(\vec{\gamma})]\urcorner \in \mathfrak{C}(\mathfrak{Ax})$ but also that $\ulcorner\forall\vec{\gamma}[\delta_2(\vec{\gamma}) \rightarrow \delta_1(\vec{\gamma})]\urcorner \in \mathfrak{C}(\mathfrak{Ax})$ because this time our ontology exhibits the third grade of semantic indeterminacy. The rest of the proof is identical to the previous case. □

As before one cannot conservatively remove the third grade of semantic indeterminacy. Although one can remove this type of indeterminacy from $\mathfrak{O}_4$ by adding to it, say, axiom 17

$$\forall x\ [\mathsf{A}(x) \rightarrow \mathsf{B}(x)], \tag{17}$$

one cannot remove it conservatively from ontology $\mathfrak{O}_5$ based on axioms 18 – 21.

$$\exists x\ [\mathsf{A}(x) \wedge \mathsf{B}(x)] \tag{18}$$

$$\exists x\ [\mathsf{A}(x) \wedge \neg\mathsf{B}(x)] \tag{19}$$

$$\exists x\ [\neg\mathsf{A}(x) \wedge \mathsf{B}(x)] \tag{20}$$

$$\exists x\ [\neg\mathsf{A}(x) \wedge \neg\mathsf{B}(x)] \tag{21}$$

4. Discussion

As the reader may expect the phenomena of semantic indeterminacy are not rare in the domain of *existing* applied ontologies. In particular, if an ontology boils down to a simple taxonomy, i.e., if it contains only formulae of the form $\forall\vec{\gamma}[\delta_1(\vec{\gamma}) \rightarrow \delta_2(\vec{\gamma})]$, then

1. it exhibits the second grade of semantic indeterminacy with respect to the set of all of its predicates;
2. it exhibits the third grade of semantic indeterminacy with respect to any set of those leaf nodes in the subsumption hierarchy that are "subsumption siblings", e.g., like set $\{\mathsf{A}, \mathsf{B}\}$ for ontology $\mathfrak{O}_1$ in section 1.[2]

However, the simple ontological widget in the form of disjointness conditions for such siblings may remove some semantic indeterminacies of the type mentioned in case 1, e.g., like the indeterminacy between predicates A and B from ontology $\mathfrak{O}_1$ in section 1. On the other hand, in order to remove the third grade of indeterminacy mentioned in case 2 one usually need to provide a non-conservative extension of the taxonomy. It is a practical experience of the author that this extension is usually realised by adding a number of relations to the ontology in question and binding the "swappable sibling" categories thereby.

These observations may provide a robust independent justification for the conviction of many ontologists to the effect that as far as the goals of applied ontology are concerned a simple taxonomy does not sufficiently characterise its categories - cf. [9,7,5,10,4]. This conviction is sometimes rendered as the slogan "ontology is not a taxonomy". Alas the number of actual ontologies that are just taxonomies is substantial.

If we restrict the domain of formal ontologies to those that are rendered in OWL, then it seems that to find out whether a given formal ontology exhibits some grade of semantic indeterminacy is a relatively easy task. In particular, we do not need any new algorithms or software tools besides those available. From the theoretical point of view it suffices to recollect the fact that the question of inferrability boils down to the question of consistency:

$$\phi \in \mathfrak{C}(X) \equiv \mathfrak{C}(X \cup \{\ulcorner \neg\phi \urcorner\}) = \mathfrak{Lang} \tag{22}$$

Consequently, all one needs to do in order to establish whether a given ontology exhibits some grade of semantic indeterminacy is to use a programming framework for OWL (e.g., Jena) coupled with an API of an OWL reasoner (e.g., Pellet). The latter will realise the required swappings and the former will check the consistency of the resulting ontologies.

[2]More precisely speaking, one can say that formal ontology $< \mathfrak{Lang}, \mathfrak{Ax} >$ is *(at least) a taxonomy* if $|\mathfrak{pred}(\mathfrak{Ax})| > 1$ and there exists exactly one predicate $\delta_0 \in \mathfrak{pred}(\mathfrak{Ax})$ such that for each other predicate $\delta \in \mathfrak{pred}(\mathfrak{Ax})$ it holds that $\ulcorner \forall\vec{\gamma}[\delta(\vec{\gamma}) \rightarrow \delta_0(\vec{\gamma})]\urcorner \in \mathfrak{C}(\mathfrak{Ax})$. Similarly, formal ontology $< \mathfrak{Lang}, \mathfrak{Ax} >$ is *(at most) a taxonomy* if for every axiom $\phi \in \mathfrak{Ax}$, there exists conjunction $\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n$ $(n > 0)$ such that (a) $\ulcorner \phi \equiv \psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n \urcorner \in \mathfrak{C}(\emptyset)$; (b) each ψ_k is equal to $\forall\vec{\gamma}\ [\delta_{k_\mathsf{a}}(\vec{\gamma}) \rightarrow \delta_{k_\mathsf{b}}(\vec{\gamma})]$; (c) and $\mathfrak{pred}(\{\phi\}) = \{\delta_{1_\mathsf{a}}, \delta_{1_\mathsf{b}}, \delta_{2_\mathsf{a}}, \ldots, \delta_{n_\mathsf{b}}\}$.

5. Implementation

Table 1 in page 12 summarises the results of a number of tests in which I investigated whether a specific applied ontology contains categories that exhibit some type of semantic indeterminacy. For obvious reasons I focused on a sample of the OWL DL ontologies that are available in the public domain - even if a given ontology has more expressive versions, I picked up its OWL DL version. Using the JENA framework (version 2.7.0) together with the Pellet reasoner (version 2.3.0) I implemented a simple "generate-and-test" algorithm to identify *couples* of named classes (instances of `owl:Class`) within these ontologies that are indiscernible as far as the axioms of those ontologies are concerned.[3] In other words, I investigated the grades of semantic indeterminacy of the ontologies in question with respect to doubletons of their (unary) categories that are rendered as named instances of `owl:Class`. Since in most cases it is not possible to list all such couples with respect to which these ontologies exhibit semantic indeterminacy, table 1 specifies only

1. examples of semantically indiscernible couples of categories from the respective ontologies;
   (a) each column on the right-hand side part of the table specifies all couples with respect to which a given ontology exhibits the *proper* first (*resp.* second, third) grade of semantic indeterminacy;
   (b) the names of categories are their local names (i.e. relative URIs);
2. ratio of such couples to all couples that one can form by means of all classes from a given ontology;
   (a) each ratio is rounded to four decimal places.

The results concerning the third grade of semantic indeterminacy require a disclaimer. Since most of the OWL ontologies use the `http://www.w3.org/2000/01/rdf-schema#label` annotation property to name its categories, these categories become *ipso facto* discernible because of those annotations. Thus, in order to implement the idea behind the notion of semantic indeterminacy I disregarded all annotations for the ontologies in question. As a result, I interpret them as comments that do not belong to the logical content of a given formal ontology.

[3]The lack of space makes it impossible to specify the details of this algorithm - the interested reader may find its JAVA implementation together with the log files for the aforementioned tests on `www.l3g.pl`. The logs specify all semantically indiscernible categories of the ontologies at stake.

Ontology		1st grade		2nd grade		3rd grade	
		Example of indiscernible couple	**Ratio**	**Example of indiscernible couple**	**Ratio**	**Example of indiscernible couple**	**Ratio**
acronym: **URI:** **version:**	BFO `http://www.ifomis.org/bfo/owl` 1.1.1	Occurrent, Quality	0.9663	-	0	Role, Disposition ProcessualContext, Process OneDimensionalRegion, ThreeDimensionalRegion	0.0337
acronym: **URI:** **version:**	CIDOC CRM `http://erlangen-crm.org/110404/` 5.0.2	E35_Title, E66_Formation	0.629	E38_Image, E56_Language	0.3707	E47_Spatial_Coordinates, E48_Place_Name	0.0003
acronym: **URI:** **version:**	DOLCE-Lite `http://www.loa-cnr.it/ontologies/DOLCE-Lite.owl` ???	physical_endurant, abstract_region	0.8934	accomplishment, state	0.1006	set, proposition accomplishment, achievement process, state dependent-place, relevant-part	0.006
acronym: **URI:** **version:**	GFO `http://www.onto-med.de/ontologies/gfo-basic.owl` 1.0 (1.13)	Symbol_structure, Presential	0.3457	Social_role, Chronoid	0.1808	Discrete_process, State Continuous_process, Discrete_process Continuous_process, State	0.0042
acronym: **URI:** **version:**	ISO 15926 `http://rds.posccaesar.org/2008/02/OWL/ISO-15926-2_2003` 2008-02-21	Classification, PossibleIndividual	0.5577	Language, IntendedRoleAndDomain	0.4412	SpatialLocation, Stream WholeLifeIndividual, ActualIndividual PeriodInTime, ActualIndividual Description, Identification	0.0011
acronym: **URI:** **version:**	PROTON KM `http://proton.semanticweb.org/2005/04/protonkm` 0.1	-	0	any couple of PROTON KM categories	1	-	0

Table 1. Semantic indeterminacy of formal applied ontologies

6. Related Work

The research topic that seems to be the most similar to the problem of semantic indeterminacy concerns ontology comparison (see [6]). Currently, the mainstream research there focuses on formal ontologies expressed in some weak description logic language, e.g. from the so-called DL-Lite family. Assume that you are to compare two "unpopulated" DL-Lite ontologies $\mathfrak{O}$ and $\mathfrak{O}'$, i.e., two sets of TBoxes/concept inclusions. Suppose also that you compare these ontologies with respect to set Σ of concepts. [6, p. 1099] defines Σ-*concept difference* between $\mathfrak{O}$ and $\mathfrak{O}'$ as the set of all concept inclusions that belong to the former but not to the latter ontology. Then ontology $\mathfrak{O}$ is said to Σ-*entail* ontology $\mathfrak{O}'$ if Σ-concept difference between $\mathfrak{O}$ and $\mathfrak{O}'$ is empty and also that $\mathfrak{O}$ and $\mathfrak{O}'$ are Σ-*inseparable* if both Σ-concept difference between $\mathfrak{O}$ and $\mathfrak{O}'$ and Σ-concept difference between $\mathfrak{O}'$ and $\mathfrak{O}$ are empty. Similarly, one can speak about Σ-query difference, Σ-query entailment, and Σ-query inseparability if one takes into account not concept inclusions but the so-called queries, i.e. formulae of the form $\exists x_1, x_2, \ldots, x_n\ \phi$, where ϕ is built out of atomic formulae and contains either variables $x_1, x_2, \ldots, x_n$, which are bounded in a query, variables $y_1, y_2, \ldots, y_n$ or the so-called object names. Finally, [6, p. 1101] defines also the notion of Σ-model difference (and its cognates) by means of the notion of Σ-model.

Although the comparison between my proposal and the aforementioned concepts from ontology comparison is not straightforward due to the obvious methodological differences of assumptions, goals, etc., some similarities are sufficiently salient. In particular, the idea of the third grade of semantic indeterminacy can be expressed in terms of Σ-concept or query entailment. Loosely speaking, an ontology exhibits the third grade of semantic indeterminacy if its "swapped" version Σ-concept or query entails it. The validity of this claim presupposes that we are interested only in DL-Lite ontologies, whose expressivity is limited to concept inclusions.[4]

Another similar research concerns the problem of unification in description logic - see, for instance, the latest survey in [1]. Briefly speaking, the latter problem deals with equivalence of concept descriptions (in the sense of description logic) under a substitution of concept descriptions for a subclass of concept names. Obviously, neither two swappable ontological categories that give rise to semantic indeterminacy of a given ontology are bound to be unifiable within this ontology nor two unifiable predicates be semantically indeterminate in the sense of definitions 1-3. However, the precise nature of the relation between the unification problem and the grades of semantic indeterminacy needs to be further investigated.

Finally ontology evaluation or, more specifically speaking, the rule-based approach to ontology evaluation - see [11] - is comparable, although rather loosely, to my proposal. For example, [3] defines the notion of ontological redundancy and provides a software application for its detection. In the terminology of my paper one can express their notion of redundancy saying that that formal ontology $< \mathfrak{Lang}, \mathfrak{Ax} >$ has a redundant axiom

[4]See the following quote from [6]:

> To simplify presentation, in this paper we do not consider DL-Lite logics with role inclusions, focusing mainly on the impact of the Boolean constructs in concept inclusions as well as number restrictions.[6, p. 1094]

ϕ – either of the form $\forall\vec{\gamma}\,[\delta_1(\vec{\gamma}) \to \delta_2(\vec{\gamma})]$ or $\delta(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n)$ (where: $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n$ are constants that are arguments of predicate δ) – if $\phi \in \mathfrak{C}(\mathfrak{Ax} \setminus \{\phi\})$. Obviously, a formal ontology with redundant axioms may or may not exhibit some grade of semantic indeterminacy and a formal ontology that exhibits some grade of semantic indeterminacy may or may not have redundant axioms.

References

[1] Franz Baader and Silvio Ghilardi. Unification in modal and description logics. *Logic Journal of the IGPL*, 19(6):705–730, 2011.

[2] Chen Chung Chang and H. Jerome Keisler. *Model theory*. Elsevier, 1990.

[3] Oscar Corcho, Asunción Gómez-Pérez, R. González-Cabero R, and María del Carmen Suárez-Figueroa. ODEval: A tool for evaluating RDF(S), DAML+OIL, and OWL concept taxonomies. In *Proceedings of the 1st IFIP Conference on Artificial Intelligence Applications and Innovations (AIAI 2004)*, pages 369–382, Toulouse, France, 2004.

[4] John Davies. Lightweight ontologies. In Poli et al. [8], pages 197–229.

[5] Elin K. Jacob. Ontologies and the Semantic Web. *Bulletin of the American Society for Information Science and Technology*, 29(4):19–22, 2005.

[6] Roman Kontchakov, Frank Wolter, and Michael Zakharyaschev. Logic-based ontology comparison and module extraction, with an application to DL-Lite. *Artificial Intelligence*, 174:1093–1141, 2010.

[7] Riichiro Mizoguchi. Tutorial on Ontological Engineering. Part 1: Introduction to Ontological Engineering. *New Generation Computing*, 21:365–384, 2003.

[8] Roberto Poli, Miecheal Healy, and Achilles Kameas, editors. *Theory and Applications of Ontology: Computer Applications*. Springer, 2010.

[9] Steffen Schulze-Kremer. Ontologies for Molecular Biology and Bioinformatics. *Silico Biology*, 2:179–193, 2002.

[10] Barry Smith, Waclaw Kusnierczyk, Daniel Schober, and Werner Ceusters. Towards a reference terminology for ontology research and development in the biomedical domain. In *KR-MED 2006 "Biomedical Ontology in Action"*, volume 222, pages 57–65, 2006.

[11] Samir Tartir, I. Budak Arpinar, and Amit P. Sheth. Ontological evaluation and validation. In Poli et al. [8], chapter 5, pages 115–130.

[12] Ryszard Wójcicki. *Theory of Logical Calculi*. Kluwer, 1988.